Technical Comments

Comments on "Derivation of Element Stiffness Matrices by Assumed Stress Distributions"

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IN the subject note,[1] and in a predecessor note,[2] Pian has provided new insight into the formulation of discrete element stiffness equations for matrix structural analysis. Although matrix methods are widely employed and have great potentialities, the scope and limitations of the component discrete element formulations are not yet completely understood. Gaps in the knowledge are being rapidly filled, however, by the forementioned references and by papers such as those by Melosh[3] and de Veubeke[4].

Reference 1 develops a procedure wherein the element boundary displacements are chosen to satisfy complete displacement compatibility, and the stress distribution throughout the element is taken to be of a form such that equilibrium is satisfied. Heretofore, the formulation of element stiffness properties has considered either assumed displacements entirely or assumed stresses entirely. The intention of this discussion is to examine closely the specific element relationships given by Pian in illustration of the new procedure and to raise certain questions pertinent to the formulation and application of element stiffness properties in general.

Consider first the assumptions used[1] for the formulation of the stiffness matrix for the rectangular plate in plane stress (Fig. 1, Ref. 1). The intent is to examine the relationship of these assumptions to the satisfaction of equilibrium and compatibility and to the assumptions employed in Ref. 5 for the formulation of the stiffness equations for the same element. The stress assumptions of Ref. 1, Eq. (20), using the first five undetermined parameters, are

$$\begin{aligned} \sigma_x &= \beta_1 + \beta_2 y \\ \sigma_y &= \beta_3 + \beta_4 x \\ \tau_{xy} &= \beta_5 \end{aligned} \tag{1}$$

Equations (1) satisfy the differential equations of equilibrium and are identical to those advanced by Turner.[5] Reference 1, however, additionally chooses linear edge displacements for use in the development of the element stiffness equations. These displacements are seemingly different in shape from those which result from integration of stress-displacement equations where the stresses are given by Eqs. (1). The general forms of the displacement equations in the latter case are[6]

$$\begin{aligned} u &= (1/E)[\beta_1 x + \beta_2 xy - \mu\beta_3 x - (\beta_4/2)(ux^2 + y^2) + \beta_6 y + \beta_7] \\ v &= (1/E)[-\mu\beta_1 y - (\beta_2/2)(x^2 + \mu y^2) + \beta_3 y + \beta_4 xy + 2(1+\mu)\beta_5 x - \beta_6 x + \beta_8] \end{aligned} \tag{2}$$

where μ is Poisson's ratio. The new parameters β_6, β_7, β_8 are constants of integration and represent the rigid body motion terms. By solution of the appropriate equations it is possible to express these displacement functions in terms of the corner point displacements:

$$\begin{aligned} u = \Big\{ &\left(1 - \frac{x}{a}\right)\left(1 - \frac{y}{b}\right) u_1 + \frac{x}{a}\left(1 - \frac{y}{b}\right) u_2 + \frac{xy}{ab} u_3 + \frac{y}{b}\left(1 - \frac{x}{a}\right) u_4 + \left[\frac{\mu x}{2b}\left(1 - \frac{x}{a}\right) + \frac{y}{2a}\left(1 - \frac{y}{b}\right)\right](v_1 - v_2 + v_3 - v_4)\Big\} \\ v = \Big\{ &\left[\frac{x}{2b}\left(1 - \frac{x}{a}\right) + \frac{\mu y}{2a}\left(1 - \frac{y}{b}\right)\right] \times (u_1 - u_2 + u_3 - u_4) + \left(1 - \frac{x}{a}\right)\left(1 - \frac{y}{b}\right) v_1 + \frac{x}{a}\left(1 - \frac{y}{b}\right) v_2 + \frac{xy}{ab} v_3 + \frac{y}{b}\left(1 - \frac{x}{a}\right) v_4 \Big\} \end{aligned} \tag{3}$$

Equations (3) contain nonlinear terms (x^2, y^2). It will be noted, however, that each equation for boundary displacement, as used in Eq. (23) of Ref. 1, represents only a linear displacement of the corresponding boundary segment in a coordinate direction, all displacements of node points other than those at the ends of the segment being held fixed. Now, it is seen that, under these conditions, Eq. (3) also represents linear edge displacement; only under a general displacement of the element nodes are the edge displacements nonlinear. Consequently, the numerical results given at the bottom of page 1335 of Ref. 1 are identical to those which would be obtained by use of the equations formulated by Turner.[5]

This fact, which has not been noted in Ref. 1, has no doubt been obscured by the foregoing considerations. The "component" matrices of Ref. 1 [Eqs. (23) and (24), etc.] are not solved to yield a single algebraic statement of the stiffness matrix.

The extremely close, if not exact, satisfaction of compatibility requirements on the part of the foregoing element relationships, coupled with satisfaction of equilibrium throughout the element, suggests that they possess particular merit. Many other rectangular plate element stiffness matrices have been proposed, however.[3, 6] Clearly, with many new concepts being advanced with considerable fervor for not only rectangles but for other geometric forms, a great deal of numerical evaluation must yet be forthcoming, i.e., "the proof of the pudding is in the tasting." The choice of efficient idealizations and element relationships remains a major problem in the field of digital computer structural analysis.

Turning now to broader questions, it is to be noted that the plate element formulation just examined is a particularly fortuitous situation in which the displacements associated with the stresses agree exactly with the independently assumed displacements of the edges. The concepts that underlie this formulation are of considerable generality and are intended by the author to apply to all forms of discrete element. Yet, in the development of new element equations, suitable stress or displacement functions are not easily found. In other cases, the algebraic difficulties associated with the use of such functions renders impractical their use in the formulation procedure. For this reason and because of guarantees on monotonic convergence, the interpolation procedures advanced by Melosh[3] appear particularly attractive.

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In routine practical analysis one must consider the relationship between economic requirements and the use of assumed functions with greater numbers of undetermined parameters than there are element node point displacements. Normally, the objective of engineering analysis is to obtain an acceptable result at minimum cost. The use of "excess" undetermined parameters may not only complicate the formulation but will certainly add expense to the computation for a given network size. Thus, it is possible that the simplest element relationships will provide acceptable results with the coarsest network one would choose to use, or else, with a larger network, at less cost. Note the closeness of the values of the terms in the stiffness matrix of Ref. 1 for 5 and 7 term stress assumptions, respectively; equations for the former, however, can be simply and explicitly formulated whereas equations for the latter are not readily formulated and otherwise require significant matrix operations.

Finally, a more complete exploitation of matrix structural analysis methods involves their use in instability, vibration, thermal stress, and inelastic analyses, among others. Consequently, it would be desirable for examinations of concepts in the formulation of discrete element relationships to include consideration of the appropriate terms for these phenomena.

References

[1] Pian, T. H. H., "Derivation of element stiffness matrices by assumed stress distributions," AIAA J. **2**, 1333–1336 (1964).

[2] Pian, T. H. H., "Derivation of element stiffness matrices," AIAA J. **2**, 576–577 (1964).

[3] Melosh, R. J., "Basis for derivation of matrices for direct stiffness method," AIAA J. **1**, 1631–1367 (1963).

[4] de Veubeke, J., "Upper and lower bounds in matrix structural analysis," *Matrix Methods of Structural Analysis* (Pergamon Press, New York, 1964), pp. 163–202.

[5] Turner, M. J., Clough, R. J., Martin, H. C., and Topp, L. J., "Stiffness and deflection analysis of complex structures," J. Aerospace Sci. **23**, 805–823 (1956).

[6] Gallagher, R. H., *A Correlation Study of Methods of Matrix Structural Analysis* (Pergamon Press, New York, 1964), Chap. 3.

Reply by Author to R. H. Gallagher

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THE author wishes to point out that Eq. (23) of Ref. 1 has been chosen to insure the boundary compatibility between neighboring elements. For example, for two neighboring rectangular elements (I) and (II) as shown in Fig. 1, the displacements $u(y)$ along the edge BC of both elements are given by the same function

$$u_{BC}(y) = [1 - (y/b)]u_B + (y/b)u_C \quad (1)$$

where u_B and u_C are the horizontal displacements at corners B and C, respectively. It is seen that the edge displacement $u_{BC}(y)$ of either element is not affected, for example, by a vertical displacement at A or F.

If the displacement functions given by Eqs. (3) of the foregoing comment is employed, two different displacement functions will be resulted. They are, on element (I),

$$u_{BC}(y) = \left(1 - \frac{y}{b}\right) u_B - \frac{y}{b} u_C + \frac{y}{2a}\left(1 - \frac{y}{b}\right) \times (v_B - v_B + v_C - v) \quad (2)$$

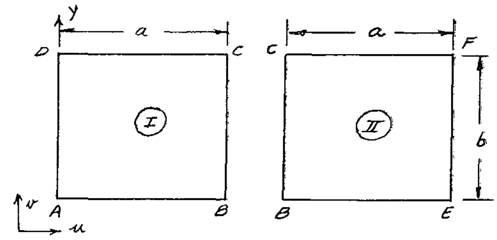

Fig. 1 Two neighboring rectangular elements.

and on element (II),

$$u_{BC}(y) = \left(1 - \frac{y}{b}\right) u_B + \frac{y}{b} u_C + \frac{y}{2a}\left(1 - \frac{y}{b}\right) \times (v_B - v_E + v_F - v_C) \quad (3)$$

Hence the two horizontal displacements will, in general, not be compatible because $(v_A - v_B - v_C - v_D)$ and $(v_B - v_E + v_F - v_C)$ are not correlated.

Thus, the author has a feeling that the numerical result given at the bottom of page 1335 of Ref. 1 are not identical to those which would be obtained by use of the equations formulated by Turner, et al., in Ref. 2.

The author is in full agreement with Gallagher that both accuracy and cost should be taken into consideration when a method is selected for a routine practical analysis, and hence there is a limit for the number of undetermined parameters to be used in formulating the stiffness matrix. The author, however, has been trying to find out whether there is basically a stiffness matrix that will yield results better than others when the same size networks are used.

References

[1] Pian, T. H. H., "Derivation of element stiffness matrices by assumed stress distributions," AIAA J. **2**, 1333–1336 (1964).

[2] Turner, M. J., Clough, R. J., Martin, H. C., and Topp, L. J., "Stiffness and deflection analysis of complex structures," J. Aeronaut. Sci. **23**, 805–823 (1956).

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Comments on "Effect of Gas Composition on the Ablation Behavior of a Charring Material"

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A PAPER by Vojvodich and Pope[1] presented the results of an experimental investigation on the ablation behavior of a charring material. Chemical reactions were shown to occur between the material and an air environment, and the gas phase reactions were found to be more significant than predicted by the theory derived from the work of Cohen, Bromberg, and Lipkis and the work of Hartnett and Eckert.[2] A close examination of the experimental results reveals a possible discrepancy in the reported combustion effects. This observation follows from a thermochemical analysis of the ablative material and a discussion and analysis reported in Ref. 3. Additional clarification of the theory[2] and notation of the similarity to the theory of Lees[4] should enhance the interpretation of the experimental results.

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